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subject is found in A. T. Bledsoe's *Philosophy of Mathematics*, Philadelphia, 1886. He holds (p. 44) that the variable never actually attains its limit, and

> ". . . this, I apprehend, will be found to be the case in relation to every variable really used in the infinitesimal method. It will, at least, be time enough to depart from the definition of Duhamel when variables are produced from the calculus which are seen to reach their limit without violating the law of their increase or decrease."

That a teacher who had pondered so long upon the foundations of the calculus as Bledsoe had done, could not think of examples of variables reaching their limits is an indication that the application of the calculus to physics and mechanics did not then receive the careful attention it deserved.

It is with the theory of limits as with negative numbers and imaginaries. In the eighteenth century it was felt that, whether such numbers could exist in algebra, was a matter of argument and demonstration; now it is merely a question of assumption. The same is true with variables reaching their limits. In modern theory it is not particularly a question of argument, but rather of assumption. The variable reaches its limit if we will that it shall; it does not reach its limit, if we will that it shall not. Our "willing" the one thing or the other consists in assuming a continuum in which the limit is a value the variable can assume; our "not willing" consists in not assuming, in the aggregate of values the variable can take, the value of the limit.

A GENERAL FORMULA FOR THE VALUATION OF BONDS.

By C. H. FORSYTH, University of Michigan.

It is the purpose of the present paper to generalize a formula for the valuation of bonds so that it will be applicable to a large number of bond offerings not satisfactorily covered by a known formula.

The problem to be considered may be put more concretely as follows. All formulas known up to the present time, cover the offering of bonds or loans only where the principal is repaid in *equal* installments. The case where payment of the principal is made in a lump sum is included, as a special case.

We shall derive a formula for computing the price of bonds where this principal is repaid in general in *unequal* installments. This formula will include cases where there may be only one installment of any one value. In fact, the formula will include as special cases not only the most general formula known at present but also all the special cases of the latter.

The most general formula[1] for valuation of bonds known up to this time is

$$k = \left(1 - \frac{a_{\overline{m(f+tr)}|} - a_{\overline{mf}|}}{r a_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{1}$$

where k represents the premium or discount—as the case may be.

The nature of the above bond offering or loan is as follows. The principal of one dollar is to be repaid in r *equal* installments, the first at the end of f years and the rest at intervals of t years. The interest named in the bond offering is g and the interest rate to be actually realized, and hence used as a basis of the annuities of the formula itself, is i, both rates to be paid m times a year. It is unnecessary to give the derivation of formula (1) here.

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[1] J. W. Glover, *A general formula for evaluating securities*, this MONTHLY, March, 1915.

Formula (1) is rarely used in the form given, much simpler formulas—special cases of (1)—being ordinarily used; but as mentioned above, it covers only those bond offerings where the repayments of the principal are *equal*. If a problem however simple, arises where the repayments are *unequal*, it is necessary to consider as many distinct offerings as there are different installments or repayments.

We propose now to derive a formula, using formula (1) as a basis, to apply to cases where these installments are, in general (but not necessarily), *unequal*.

The nature of such a bond is as follows. The principal of S dollars is to be repaid in $r_1 + r_2 + \cdots + r_n$ installments, of which the first r_1, the next r_2, etc., are equal, there being n *different* installments.

The first installment is to be paid at the end of f_1 years, the next $r_1 - 1$ at intervals of t_1 years, the first of the r_2 payments at the end of f_2 years (from the date of the whole bond offering), the next $r_2 - 1$ at intervals of t_2 years, etc., and if we designate as " major intervals " those intervals wherein all the installments are equal, we may say in general that in the nth major interval there will be r_n installments at intervals of t_n years, the first of which is to be paid at the end of f_n years.

The rate of interest offered in the loan is g and the rate to be actually realized is i, both to be paid m times a year.

We shall let S_1 represent the sum of the r_1 equal installments of the first major interval, and in general S_n the sum of the r_n equal installments of the nth major interval; and we shall also let I_n represent the value of each of these installments, that is put I_n equal to S_n/r_n.

Then applying formula (1) itself, considering the several major intervals as so many different bond offerings, the value of the premium or discount on the whole bond offering becomes at once

$$\begin{aligned} k = {} & \frac{S_1}{S}\left(1 - \frac{a_{\overline{m(f_1+r_1t_1)}|} - a_{\overline{mf_1}|}}{r_1 a_{\overline{mt_1}|}}\right)\left(\frac{g-i}{i}\right) \\ & + \frac{S_2}{S}\left(1 - \frac{a_{\overline{m(f_2+r_2t_2)}|} - a_{\overline{mf_2}|}}{r_2 a_{\overline{mt_2}|}}\right)\left(\frac{g-i}{i}\right) \\ & \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ & + \frac{S_n}{S}\left(1 - \frac{a_{\overline{m(f_n+r_nt_n)}|} - a_{\overline{mf_n}|}}{r_n a_{\overline{mt_n}|}}\right)\left(\frac{g-i}{i}\right). \end{aligned}$$

Adding, we have

$$k = \left(1 - \frac{\sum_1^n a_n}{S}\right)\left(\frac{g-i}{i}\right), \tag{2}$$

where we let $A_n = I_n/a_{\overline{mt_n}|}$ and $a_n = A_n(a_{\overline{m(f_n+r_nt_n)}|} - a_{\overline{mf_n}|})$.

Ordinarily we might expect the first interval to be the same as the other intervals of any major interval. However, it should be emphasized that nothing has been assumed to prevent this first interval from being any number of years. In other words, we may, if we choose, look upon the first installment of each major interval as being deferred any arbitrary number of years just as it is openly assumed in the first major interval. This period of deferment may, of course, take as a special case the value of each of the other intervals (t_n in the nth major interval) of that major interval. If, in this special case, we have also $t_1 = t_2 = \cdots = t_n = t$, then

$$f_1 = f,\ f_2 = f + r_1 t,\ f_3 = f + (r_1 + r_2)t,\ \cdots,\ f_n = f + (r_1 + r_2 + \cdots + r_{n-1})t$$

and formula (2) becomes

$$k = \left(1 - \frac{\sum_0^n (I_n - I_{n+1})a(n)}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{3}$$

where $a(n) = a_{\overline{m[f+(r_1+r_2+\cdots+r_n)t]}|}$, $a(0) = a_{\overline{mf}|}$, and $I_0 = I_{n+1} = 0$.

Formula (3) appears more complicated than (2) but on practical application will prove much simpler and, of course, much less general. One must be careful in assigning the proper values to the f's in a practical problem in using formula (2). In general, f_n extends from the date of the whole bond offering till the first installment of the nth major interval.

As an example and special case where we have only two major intervals or only two different installments, formula (3) may be expanded into the form

$$k = \left(1 - \frac{a(2)I_2 + a(1)(I_1 - I_2) - a(0)I_1}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right), \tag{4}$$

and so on, in general, for n equal to any positive integer.

Perhaps we should add that in case any major interval should contain only one installment (one of the r's becomes unity) it may be asked what value should be given the corresponding value of t for that major interval. On investigation one will find that in that case the troublesome terms involving that particular t will cancel out in the formula and hence prove immaterial in value in practice. However, we must be careful to not give it the value zero, for in that one case an indeterminate expression will arise and make unnecessary trouble. We would suggest that the simplest plan in practice would be to assume the value of that particular t to be unity, as that value would make a minimum of trouble and will not necessitate the previous and troublesome cancellation of the terms in t.

If we assume n to be unity or only one value for the values of the installments, our formula (3) reduces to

$$k = \left(1 - \frac{a(1)I_1 - a(0)I_1}{Sa_{\overline{mt}|}}\right)\left(\frac{g-i}{i}\right),$$

which is identically the well-known formula (1).

It should be added that all annuities are to be valued, as formerly with formula (1), at the rate of interest i/m. Formulas (2) and (3) prove very easy to apply in a concrete problem and the values of the annuities involved are easily determined. We shall conclude with a typical though simple example. Let it be desired to find the value or price of a loan of \$50,000 offered at 6 per cent. and dated September 1, 1914; the principal to be repaid in fifteen installments, \$2,000 September 1, each year from 1919 to 1928 and \$6,000 September 1, from 1929 to 1933, to net the purchaser 5 per cent.

Here we have $f = 5$, $m = t = 1$, $g = .06$, $i = .05$ and $n = 2$ and we can use formula (3) as simplified and expressed in formula (4), whence

$$k = \left(1 - \frac{a(2)\cdot 6000 + a(1)(2000 - 6000) - a(0)\cdot 2000}{50{,}000 a_{\overline{1}|}}\right)\left(\frac{.06 - .05}{.05}\right),$$

where

$$a(2) = a_{\overline{m(f+r_1t+r_2t)}|} = a_{\overline{20}|}$$

$$a(1) = a_{\overline{m(f+r_1t)}|} = a_{\overline{15}|}$$

$$a(0) = a_{\overline{mf}|} = a_{\overline{5}|},\ \text{all to be valued at 5 per cent.}$$

The values of these annuities can be obtained from any set of tables. Hence, finally, the premium is $k = .0966977$, and the price of the entire loan becomes \$50,000 $(1 + k)$ or \$54,834.89.

A GEOMETRICAL INTERPRETATION OF GREEN'S FORMULA.

By W. V. LOVITT, Purdue University.

It is the object of this note to give a geometrical interpretation of Green's formula:

$$\int_{(C)} P(x, y)dx + Q(x, y)dy = \int_A\int\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dxdy.$$

The single integral is to be taken in the positive direction around a closed curve

$$(C) \qquad C(x, y) = 0$$

in the xy-plane and the double integral over the interior A of C. To derive this formula it is sufficient to prove the formulas

$$(1) \qquad \int_A\int \frac{\partial P}{\partial y}dxdy = -\int_{(C)} Pdx,$$

$$(2) \qquad \int_A\int \frac{\partial Q}{\partial x}dxdy = \int_{(C)} Qdy$$

(see for example Goursat-Hedrick, § 126), and for simplicity it may be assumed